COBORDISM THEORY AND LOCALIZATION FORMULAS FOR HAMILTONIAN GROUP ACTIONS

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In this note we will report on some symplecto-geometric[1] applications of the following result:

Theorem 1. *Let M be a compact oriented $2d$-dimensional manifold on which the group S^1 acts. Suppose that this action is quasi-free and has finitely many fixed points. Then M is cobordant to a disjoint union of N copies of $\mathbb{CP}^d$, where N is the number of fixed points.*

A more detailed discussion of this theorem and its applications will appear in [GGK].[2]

Remark. "Quasi-free" means that S^1 acts freely on the complement of the fixed point set. The cobordism is an S^1-equivariant cobordism with the $\mathbb{CP}^d$'s carrying projective S^1-actions. Furthermore, the cobordism is oriented although the $\mathbb{CP}^d$'s need not be given their standard orientations.

Proof. The circle S^1 acts on $M \times \mathbb{C}$ by the product of its action on M and its standard action on $\mathbb{C}$. The fixed points of this action are

$$q_k = (p_k, 0) \ , \ p_k \in M^{S^1} \ ,$$

where M^{S^1} is the set of fixed points for the circle action on M. Denote by U_k an S^1-invariant open ball around q_k (with respect to some invariant Riemann metric). Let W be the subset of $M \times \mathbb{C}$ obtained by excising the U_k's and the set $|z| > 1$. Since S^1 acts freely on W, the quotient W/S^1 is a compact manifold-with-boundary, and, modulo

orientations,

$$\partial(W/S^1) = M \cup \coprod_k (\partial U_k)/S^1 . \tag{1}$$

Let T_{q_k} be the tangent space of $M \times \mathbb{C}$ at q_k. The linear isotropy action of S^1 on T_{q_k} is free except at the origin, hence there is an $\mathbb{R}$-linear identification

$$T_{q_k} \cong \mathbb{C}^{d+1} \tag{2}$$

which converts this action into the action "multiplication by $e^{i\theta}$". Via (2) one can identify U_k with the set $||z|| < \varepsilon$ and hence identify $\partial U_k/S^1$ with $\mathbb{CP}^d$. Thus, by (1), M is cobordant to the disjoint union of N copies of $\mathbb{CP}^d$.

If the isomorphism (2) respects orientation, the k-th $\mathbb{CP}^d$ is equipped with the complex orientation; otherwise we take the opposite orientation.

The action of S^1 on the first component of $M \times \mathbb{C}$ commutes with the diagonal action and thus descends to W/S^1, making it into an equivariant cobordism.   Q.E.D.

In what follows we will exploit the fact that certain structures on M which are preserved by the S^1-action (for instance, a closed 2-form, a stable complex structure, or a G-action) can be incorporated into the cobordism (1). For example, we have already seen how the cobordism inherits the S^1-action. Before introducing these new structures, though, we would like to remark that the assumptions made in Theorem 1 are unrealistically strong. What happens if the action of S^1 on M is not quasi-free, or does not have finitely many fixed points? If the action is not quasi-free, the induced action on W is locally free, but not free, so the cobording space W/S^1 is now an *orbifold*. Moreover, under the identification (2), the S^1-action on $\mathbb{C}^{d+1}$ has the form

$$e^{i\theta}z = (e^{im_1\theta}z_1, \dots, e^{im_{d+1}\theta}z_{d+1}). \tag{3}$$

Here the m_j's are nonzero integers which are no longer equal to ± 1, and the quotient, $\partial U_k/S^1$, is also an orbifold (a twisted projective space). Therefore, from (1) one gets an equivariant *orbifold* cobordism between M and a disjoint union of twisted projective spaces. If, on the other hand, the action is quasi-free but the fixed point set is not finite, one can prove:

Theorem 2. *Let X_k, $k = 1, \dots N$, be the connected components of the fixed point set M^{S^1}. Then there is an equivariant cobordism*

$$M \sim \coprod_{k=1}^{N} B_k \tag{4}$$

and fibrations

$$\mathbb{CP}^{m_k} \hookrightarrow B_k \to X_k \tag{5}$$

where $2m_k = \operatorname{codim} X_k$.

Finally, if neither of these hypotheses holds, there exists an orbifold cobordism of the form (4); however, the $\mathbb{CP}^{m_k}$'s in (5) have to be replaced by twisted projective spaces.

The main result of this article is a reformulation of Theorem 1 in the setting of Hamiltonian group actions. To state this result we will first explain what is meant by "cobordism" in this setting.

Definition. Let G be a compact Lie group and let (M_r, ω_r) be a compact oriented[3] $2d$-dimensional symplectic manifold on which G acts in a Hamiltonian fashion, with $\phi_r \colon M_r \to \mathfrak{g}^*$ as the associated moment mapping. One says that (M_1, ω_1, ϕ_1) and (M_2, ω_2, ϕ_2) are cobordant as Hamiltonian G-spaces if there exists a compact oriented $(2d + 1)$-dimensional manifold-with-boundary W, a closed two-form ω, and a Hamiltonian action of G with moment map $\phi \colon W \to \mathfrak{g}^*$, such that

$$\partial W = M_1 \cup (-M_2)$$

and such that the pull-backs of ω and ϕ to M_r are ω_r and ϕ_r.

Remarks.
  1. This definition is different from the definition used in [Gin] in that we do *not* impose any rank or non-degeneracy condition on the closed form ω.
  2. By a "Hamiltonian space" we will generally mean a manifold equipped with a group action, a closed invariant two-form (not necessarily symplectic), and a corresponding moment map. Our definition of cobordism extends word-for-word to the case where the two-forms ω_r are degenerate, and therefore we have the notion of cobordism of Hamiltonian spaces.
  3. The equivariant form $\omega_r + \phi_r$ can be altered by an equivariant coboundary without changing the cobordism class of the Hamiltonian space (M_r, ω_r, ϕ_r), so the cobordism class of a Hamiltonian

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[3] N.B. The orientation of M need *not* be identical with its symplectic orientation (see [Gin]).

G-space depends only on the underlying G-manifold and on the cohomology class $[\omega_r + \phi_r]$.
4. In all our applications the group G will be a torus.

Apropos of this definition we have proved [GGK] the following:

Lemma 1. *Let T be a torus and let (W, ω, ϕ) be a cobordism of Hamiltonian T-spaces between (M_r, ω_r, ϕ_r), $r = 1, 2$. Assume that a is a regular value of ϕ_r for $r = 1, 2$. Then one can perturb (ω, ϕ) away from the boundary of W so that a becomes a regular value of ϕ.*

A consequence of this is that *cobordism commutes with reduction.* More explicitly, let $Z = \phi^{-1}(a)$. The fact that a is a regular value of ϕ implies that the action of T on Z is locally free, hence Z/T is an orbifold-with-boundary. There exists a closed two-form ω^{red} on Z/T whose pull-back to Z is equal to the restriction to Z of ω. The boundary components of Z/T are the reduced spaces M_r^{red}, and the restrictions of ω^{red} to each of these boundary components is the usual reduced symplectic form. All of this remains true even when ω is degenerate, and implies:

Theorem 3. *Let a be a regular value of ϕ_r for $r = 1, 2$. Then the reduced spaces $M_r^{\text{red}} = \phi_r^{-1}(a)/T$ are cobordant (as symplectic orbifolds).*

This theorem indicates that, when working with Hamiltonian group actions, orbifold cobordism is a far more natural notion than the usual notion of cobordism. From now on we will tacitly allow all our cobordisms to be orbifold cobordisms. We would like to note, however, that not every orbifold is cobordant to a manifold (even over the rationals), and in passing to orbifolds we essentially enlarge the set of cobordism classes [GGK].

Before stating our main result we will make a few remarks about cobordisms between *non*-compact manifolds: let (M, ω, ϕ) be a $2d$-dimensional Hamiltonian S^1-space with moment map ϕ. As in the proof of Theorem 1 we will let S^1 act on $M \times \mathbb{C}$ diagonally, by the product of its action on M and its standard action on $\mathbb{C}$. If one equips $\mathbb{C}$ with the two-form $\sqrt{-1}\, dz \wedge d\overline{z}$, this becomes a Hamiltonian action with moment map

$$\psi(m, z) = \phi(m) + |z|^2 \ .$$

It is clear that a is a regular value of ϕ if and only if a is a regular value of ψ, in which case one can reduce $M \times \mathbb{C}$ at a to obtain an orbifold,

$$M^a := \psi^{-1}(a)/S^1 \ .$$

Lemma 2. *The orbifold M^a is the disjoint union of the set*

$$M_{\phi<a} = \{p \in M, \phi(p) < a\}$$

and the reduced space $M_a = \phi^{-1}(a)/S^1$.

Proof. A pair (m, z) is in $\psi^{-1}(a)$ if and only if $\phi(m) + |z|^2 = a$. Let $z \neq 0$. Then $\phi(m) < a$ and

$$z = e^{i\theta}(a - \phi(m)), \ \theta = \arg z \ .$$

On the other hand, when $z = 0$, we have $\phi(m) = a$, and the S^1-orbit through (m, z) can be identified with a point of M_a.   Q.E.D.

In particular, if ϕ is proper and bounded from below, M^a is compact and hence is an orbifold compactification of the open subset $M_{\phi<a}$ of M. Notice also that, as above, the product of the action of S^1 on M with the trivial action on $\mathbb{C}$ commutes with the action we have just described and hence induces a Hamiltonian action of S^1 on M^a. Moreover, if a compact Lie group G acts in a Hamiltonian fashion on M and this action commutes with the action of S^1, one gets an induced Hamiltonian action of G on M^a.

The operation

$$M \mapsto M^a$$

is called *symplectic cutting*. For more details about it see [Ler].

Now let (M_r, ω_r), for $r = 1, 2$, be a Hamiltonian S^1-space with moment map $\phi_r \colon M_r \to \mathbb{R}$. We will assume that ϕ_r is proper and bounded from below.

Definition. The orbifolds M_1 and M_2 are cobordant as Hamiltonian S^1-spaces if the cut spaces M_1^a and M_2^a are cobordant as Hamiltonian S^1-spaces for all values of a.

Suppose that, in addition, there is a Hamiltonian action of G on M_r which commutes with the action of S^1. Then we say that M_1 and M_2 are cobordant as Hamiltonian G-spaces if M_1^a and M_2^a are cobordant as Hamiltonian G-spaces for all a.

Remarks.

1. Although M_r might be non-compact, the condition on the moment map ϕ_r guarantees that the cut space M_r^a is compact. The cobording manifold between these cut spaces is required to be compact too.

2. If M_1^a is cobordant to M_2^a for some value a then M_1^b is cobordant to M_2^b for all $b < a$; this cobordism is obtained by taking the cobording manifold with boundary $M_1^a \sqcup (-M_2^a)$, and cutting it at the value b.

3. One might hope to find a single (possibly non-compact) cobordism W between M_1 and M_2; its cuts, W^a, would then be cobordisms between M_1^a and M_2^a. However, this is not always possible.

Let (M, ω, ϕ) be a Hamiltonian S^1-space with moment map $\phi \colon M \to \mathbb{R}$. We will assume that ϕ is proper and bounded from below. Also, for the moment we will assume that the fixed point set is discrete; $M^{S^1} = \{p_1, p_2, \dots\}$. For each fixed point p_k, there is an $\mathbb{R}$-linear orientation preserving map

$$T_{p_k} \cong \mathbb{C}^d \tag{6}$$

which converts the isotropy action of S^1 into the action

$$e^{i\theta} z = (e^{im_{1k}\theta} z_1, \dots, e^{im_{dk}\theta} z_d). \tag{7}$$

If ω is symplectic, we can assume that the isomorphism (7) converts the symplectic form on T_{p_k} into the standard symplectic form on $\mathbb{C}^d$:

$$\frac{\sqrt{-1}}{2} \sum dz_r \wedge d\overline{z}_r \ . \tag{8}$$

In any case, let ω_k be the symplectic form

$$\omega_k = \frac{\sqrt{-1}}{2} \sum \epsilon_{rk} dz_r \wedge d\overline{z}_r \tag{9}$$

where $\epsilon_{rk} = \operatorname{sgn}(m_{r,k})$, and let σ_k be the integer

$$\sigma_k = \#\{m_{rk}, \ \epsilon_{rk} = -1\}. \tag{10}$$

The action (7) is Hamiltonian with respect to both the forms above; however, the associated moment maps are different. The moment map associated with the form (8) is

$$\frac{1}{2} \sum m_{rk} |z_k|^2 \quad \text{(plus an additive constant)},$$

and the moment map associated with the form (9) is

$$\frac{1}{2} \sum \epsilon_{rk} m_{rk} |z_k|^2 \quad \text{(plus an additive constant)}.$$

Since $\epsilon_{rk} m_{rk} = |m_{rk}| > 0$, the second of these maps is bounded from below and proper.

We can now state our main result.

Theorem 4 (Linearization theorem). *M is cobordant, as a Hamiltonian S^1-space, to the disjoint union of the linear spaces $(\mathbb{C}^d, \omega_k)$, $k = 1, 2, \ldots$, where $(\mathbb{C}^d, \omega_k)$ is equipped with the Hamiltonian action of S^1 defined by (7), with the moment map*

$$\phi_k = \frac{1}{2} \sum \epsilon_{rk} m_{rk} |z_k|^2 + \phi(p_k) , \tag{11}$$

and with its complex orientation.

Note that the complex orientation on $\mathbb{C}^d$ is equal to $(-1)^{\sigma_k}$ times the symplectic orientation induced by $(\omega_k)^d$.

Remarks.

1. If, in addition, a compact Lie group G acts on M in a Hamiltonian fashion, and if this action commutes with the action of S^1, then this cobordism will be a cobordism of Hamiltonian G-spaces. (For instance, this will be the case if S^1 is a subgroup of the center of G.) This shows that the circle in Theorem 4 can be replaced by a torus acting on M with isolated fixed points.

2. If the set of fixed points is not discrete, an analogue of Theorem 4 is true with the T_{p_k}'s replaced by the normal bundles to the fixed point components; for details see [GGK].

3. If ω is not symplectic and M is oriented, the isotropy weights m_{rk} for every fixed point p_k are only determined up to a simultaneous change of sign of an even number of them. However, the integers σ_k and $\epsilon_{rk} m_{rk}$ are well defined and Theorem 4 remains true for any choice of m_{rk}'s.

The rest of this article is devoted to applications of Theorem 4. However, before getting into details, let us explain the main idea of how to use cobordisms to evaluate certain invariants of Hamiltonian spaces. The cohomological invariants such as, for example, the Duistermaat-Heckman measure are, by Stokes's theorem, invariants of cobordism. This is also true for the cobordism class of the symplectic reduction and for the (equivariant) Riemann-Roch number, i.e., the virtual geometric quantization. (In the latter case, the notion of cobordism is to be modified to take into account the stable complex structure of the symplectic manifold; see Application 4 below.) By Theorem 1 (or Theorem 4), a Hamiltonian space M with isolated fixed points is cobordant to a disjoint union of twisted projective spaces (or linear spaces) associated with the local data near the fixed point set. Therefore, a cobordism invariant of M can be expressed as the sum of invariants of these spaces. For example, when dealing with the Liouville measure, this procedure

leads immediately to the Duistermaat-Heckman formula (Application 2).

Application 1 – Symplectic Reduction

Suppose that a torus T acts in a Hamiltonian fashion on (M, ω) with isolated fixed points $\{p_k\}$. Let $\phi \colon M \to \mathfrak{t}^*$ be the moment map associated with this action and let $\phi_k \colon \mathbb{C}^d \to \mathfrak{t}^*$ be the moment map associated with the linear isotropy action (7) of T on $T_{p_k} \cong \mathbb{C}^d$ and with the two-form ω_k which is given by (9). This map is unique up to an additive constant and we will fix this constant by requiring that $\phi_k(0) = \phi(p_k)$.

Suppose now that $a \in \mathfrak{t}^*$ is a regular value of ϕ and of the ϕ_k's. Then, by Theorem 3, the reduced space

$$M_{\mathrm{red}} = \phi^{-1}(a)/T \tag{12}$$

with its reduced two-form is cobordant to the disjoint union of

$$M_k = \phi_k^{-1}(a)/T \ , \qquad k = 1, 2, \ldots . \tag{13}$$

These spaces are compact symplectic toric orbifolds (see [LT]). This proves:

Theorem 5. M_{red} *is cobordant to a disjoint union of compact symplectic toric orbifolds.*

This result was also proved by Shaun Martin [Mar]; he expressed M_{red} as a "tower" of twisted projective spaces (i.e., a bundle over a bundle over ... etc., where the fibers are twisted projective spaces).

Remark. Theorem 5 can be modified to also cover the case where a is a regular value for ϕ but not a regular value for the ϕ_k's. In this case, M_{red} is still cobordant to a disjoint union of compact toric orbifold, but the two-forms on these might be degenerate. For details, see [GGK].

Application 2 – The Duistermaat-Heckman theorem

Let T be a torus and let (M, ω, ϕ) be a compact Hamiltonian T-space with isolated fixed points. Let $(\mathbb{C}^d, \omega_k, \phi_k)$ be as in Theorem 4.

Let μ be the measure on Borel subsets of M associated with the top-form $\omega^d/d!$, and let μ_k be the measure on $\mathbb{C}^d$ associated with $(\omega_k)^d/d!$. Their push-forwards,

$$\nu = \phi_* \mu \quad \text{and} \quad \nu_k = (\phi_k)_* \mu_k,$$

are the Duistermaat-Heckman measures associated with the Hamiltonian spaces (M, ω, ϕ) and $(\mathbb{C}^d, \omega_k, \phi_k)$. Theorem 4, coupled with the

fact that the Duistermaat-Heckman measure is a cobordism invariant
of Hamiltonian T-spaces, implies that

$$\nu = \sum_k \nu_k \ . \tag{14}$$

Note that μ and μ_k are *signed* measures, even in the symplectic case;
the top form $(\omega_k)^d/d!$ might be negative with respect to the (com-
plex) orientation of $\mathbb{C}^d$. If, instead, we integrate with respect to the
symplectic orientations, we get the more familiar formula of Guillemin-
Lerman-Sternberg:

$$\nu = \sum_k (-1)^{\sigma_k} |\nu_k|$$

which involves the positive push-forward measures $|\nu_k|$.

This identity was proved by other means in [GLS] (for compact
spaces) and in [PW] (for certain non-compact Hamiltonian spaces).

APPLICATION 3 – THE JEFFREY-KIRWAN LOCALIZATION THEOREM

Let us show how to use cobordisms to obtain the Jeffrey-Kirwan lo-
calization theorem in the abelian case. By Application 1, the reduction
M_{red} is cobordant to a disjoint union of toric varieties M_k associated
with the fixed points of the action. Therefore, for a cohomology class
$c \in H_T^*(M)$, the integral of the restriction of c to M_{red} is equal to
the sum of integrals of its restrictions to the M_k's. This is in fact the
Jeffrey-Kirwan localization theorem, as long as we do not care how to
carry out the integration over M_k's explicitly.

Let us get more specific. For a regular value a of the moment map
ϕ, there is a canonical map (the Kirwan map)

$$\kappa_a \colon H_T^*(M) \to H^*(M_{\text{red}})$$

which maps the equivariant cohomology of M (with coefficients in $\mathbb{C}$)
surjectively onto the ordinary cohomology of the reduced space $M_{\text{red}} =
\phi^{-1}(a)/T$. By definition, this map is the composite of the restriction
mapping

$$H_T^*(M) \to H_T^*(\phi^{-1}(a))$$

and the Cartan isomorphism

$$H_T^*(\phi^{-1}(a)) \cong H^*(\phi^{-1}(a)/T) \ .$$

Also, for each p_k, one has a canonical homomorphism,

$$\sigma_k \colon H_T^*(M) \to H_T^*(\{p_k\}) \overset{\cong}{\to} H_T^*(\{0\}) \cong H_T^*(\mathbb{C}^d)$$

the first arrow being the restriction map. In the term on the right, the action of T on $T_{p_k} = \mathbb{C}^d$ is the isotropy action given by (7). In addition, there is a Kirwan map

$$\kappa_k \colon H_T^*(\mathbb{C}^d) \to H^*(M_k)$$

where M_k is the reduced space (13), a toric variety. Applying Stokes's theorem to the cobordism described in Theorem 5, one can deduce, for $c \in H_T^*(M)$, the following identity:

$$\int_{M_{\mathrm{red}}} \kappa(c) = \sum_k \int_{M_k} \kappa_k \circ \sigma_k(c) \ . \tag{15}$$

Remarks.

1. This is a topological form of the Jeffrey-Kirwan localization theorem ([JK]). Their version of this theorem is valid for nonabelian groups. However, Shaun Martin [Mar] has recently given a purely topological proof that the abelian version of the localization theorem implies the non-abelian version.
2. An explicit recipe for evaluating the terms on the right of (15) is given in [GS].
3. We have implicitly assumed that a is a regular values for the ϕ_k's. This assumption can be avoided [GGK].
4. As before, if we equip the M_k's with their *symplectic* orientations, the summands of (15) need to be taken with the coefficients $(-1)^{\sigma_k}$.

Application 4 – Quantization

A *stable complex structure* on a manifold M is, by definition, a complex structure on the bundle $TM \oplus \mathbb{R}^\ell$ for some ℓ. Two such structures, one on the bundle $TM \oplus \mathbb{R}^{\ell_1}$ and one on the bundle $TM \oplus \mathbb{R}^{\ell_2}$, are said to be equivalent if there exists, for some choice of m_1 and m_2, an isomorphism of complex vector bundles

$$TM \oplus \mathbb{R}^{\ell_1} \oplus \mathbb{C}^{m_1} \cong TM \oplus \mathbb{R}^{\ell_2} \oplus \mathbb{C}^{m_2} \ .$$

For example, an almost complex structure on M can always be viewed as a stable complex structure. Furthermore, on a symplectic manifold (M, ω) there is a canonical stable complex structure. Namely on TM itself there exists a complex structure which is compatible with ω and this complex structure is unique up to isomorphism. If ω is invariant with respect to the action of a compact group G, the almost complex structure can also be chosen G-invariant.

Given two compact oriented manifolds M_r, $r = 1, 2$, each equipped with a stable complex structure J_r, one says that (M_1, J_1) and (M_2, J_2) are *cobordant* if there exists a compact oriented manifold-with-boundary W and a stable complex structure J on W such that

$$\partial W = M_1 \cup (-M_2) \tag{16}$$

and

$$\iota_r^* J \sim J_r, \tag{17}$$

$\iota_r \colon M_r \to \partial W$ being the inclusion mapping. (The equivalence (17) makes sense in view of the fact that $\iota_r^*(TM \oplus \mathbb{R}^\ell) = TM_r \oplus \mathbb{R}^{\ell+1}$.) With these definitions one has the following addendum to Theorem 4.

Theorem 6. *Let us equip TM with an invariant complex structure J and choose isomorphisms (6) which respect this structure. Then the cobordism described in Theorem 4 is a cobordism of stable complex structures, i.e., we have*

$$(M, \omega, \Phi, J) \sim \sum_k (\mathbb{C}^d, \omega_k, \phi_k, i) \ , \tag{18}$$

where i denotes the intrinsic complex structure on $\mathbb{C}^d$.

Remark. As before, if on $(\mathbb{C}^d, \omega_k)$ we take the symplectic orientation instead of the complex orientation, we must put the coefficient $(-1)^{\sigma_k}$ in front of it.

Recall that the cobordism between M and the $\mathbb{C}^d$'s described in Theorems 4 and 6 is, strictly speaking, a cobordism between spaces which are obtained from these by the "symplectic cutting" operation. Therefore, one item which remains to be explained in the statement of Theorem 6 is how the stable complex structures on M and on the $\mathbb{C}^d$'s give rise to stable complex structures on the spaces obtained from them by symplectic cutting. Since symplectic cutting is just a special case of symplectic reduction, the answer to this is provided by the following:

Theorem 7. *Let T be a torus and M a Hamiltonian T-space. Fix a regular value a of the moment map $\phi \colon M \to \mathfrak{t}^*$. Then a T-invariant stable complex structure on M induces a stable complex structure on the reduced space $M_{\mathrm{red}} = \phi^{-1}(a)/T$.*

Proof. Letting $Z = \phi^{-1}(a)$ and letting $\iota \colon Z \to M$ and $\pi \colon Z \to M_{\mathrm{red}}$ be the inclusion and projection maps, it is easy to see that

$$\iota^* TM = \pi^* TM_{\mathrm{red}} \oplus (\mathfrak{t} \oplus \mathfrak{t}^*) \ . \tag{19}$$

Thus a T-invariant complex structure on $TM \oplus \mathbb{R}^\ell$ induces a T-invariant complex-structure on $\pi^* TM_{\mathrm{red}} \oplus (\mathfrak{t} \oplus \mathfrak{t}^*) \oplus \mathbb{R}^\ell$. Since T is abelian, the action of T on $\mathfrak{t} \oplus \mathfrak{t}^*$ is trivial, so this is equivalent to a complex structure on $TM_{\mathrm{red}} \oplus (\mathfrak{t} \oplus \mathfrak{t}^*) \oplus \mathbb{R}^\ell$.   Q.E.D.

Given a compact manifold M with a stable complex structure J and a complex line bundle $\boldsymbol{L} \to M$, one defines the *Riemann-Roch number* of $(M, J, \boldsymbol{L})$ to be the integral

$$\int_M \exp c(\boldsymbol{L}) \mathrm{Todd}(M, J) \ , \tag{20}$$

$\mathrm{Todd}(M, J)$ being the Todd class of the complex vector bundle $TM \oplus \mathbb{R}^\ell$ and $c(\boldsymbol{L})$ being the Chern class of $\boldsymbol{L}$. One can also define an equivariant version of (20) (cf. [BGV]) and an orbifold version of (20) (the Kawasaki Riemann-Roch number of $(M, J, \boldsymbol{L})$, cf. [Kaw].)[4]

Suppose now that (M, ω) is a pre-quantizable Hamiltonian T-space with a pre-quantum line bundle $\boldsymbol{L}$. Assume for the moment that ω is symplectic.[5] If $a \in \mathfrak{t}^*$ is an integer lattice point then the reduced space (12) is pre-quantizable. Let $\boldsymbol{L}_{\mathrm{red}}$ be its pre-quantum line bundle, J_{red} an almost complex structure on M_{red} which is compatible with its symplectic structure, and $\mathrm{RR}(M_{\mathrm{red}}) = \mathrm{RR}(M_{\mathrm{red}}, J_{\mathrm{red}}, \boldsymbol{L}_{\mathrm{red}})$. Similarly, for each fixed point p_k, let M_k be the reduced space (13) and $\mathrm{RR}(M_k) = \mathrm{RR}(M_k, J_k, \boldsymbol{L}_k)$ where $\boldsymbol{L}_k$ is the reduced pre-quantum line bundle and J_k is the stable complex structure associated via Theorem 7 with the intrinsic complex structure on $\mathbb{C}^d$. (Usually this will *not* be compatible with the symplectic form on M_k!) Applying the Jeffrey-Kirwan theorem to the equivariant version of $c(\boldsymbol{L}) \mathrm{Todd}(M, J)$, one gets the following "quantization" identity:

$$\mathrm{RR}(M_{\mathrm{red}}) = \sum_k \mathrm{RR}(M_k) \tag{21}$$

expressing the Riemann Roch number of the reduced space in terms of Riemann-Roch numbers of toric orbifolds.

Remark. A stable complex structure J_k does *not* determine an orientation. The orientation on M_k which we take in (21) is induced by the complex orientation of $\mathbb{C}^d$; see [CKT] for details. If, instead, we take the symplectic orientation determined by ω_k, we must insert the coefficients $(-1)^{\sigma_k}$ in front of the summands in (21).

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[4] Warning: the formula for the Kawasaki Riemann-Roch number is more complicated than (20).

[5] A detailed treatment of quantization in the case that the two-form ω is degenerate will appear in [CKT].

Note that, in principle, $\mathrm{RR}(M_k)$ only depends on the linear isotropy representation of T on T_{p_k}, though in practice its evaluation relies on some rather deep theorems of Brion-Vergne [BV], Cappell-Shaneson [CS] and Guillemin [Gu]. In [GGK] we will give an explicit formula for it in terms of a partition function involving the weights of this representation.

Stable complex structures can be incorporated in the notion of cobordism of Hamiltonian G-spaces to make the geometric quantization, i.e., the equivariant Riemann-Roch number, into an invariant of cobordism. Thus consider the category of formal Hamiltonian G-spaces, i.e., oriented G-manifolds (or orbifolds) equipped with an equivariant closed 2-form $\omega + \phi$ (or just its equivariant cohomology class) and a G-equivariant stable complex structure J. Clearly, an oriented symplectic G-manifold with a fixed moment map can be viewed as a formal Hamiltonian G-space. An example of a different nature is the $\mathbb{C}^d$'s from Theorem 6, where the symplectic form need not be compatible with the complex structure. Two such manifolds or orbifolds are said to be *strongly* cobordant if they are cobordant as Hamiltonian G-spaces and the cobordism W can be chosen to carry a G-equivariant stable complex structure which extends the structures on its boundary. Using symplectic cutting one can extend this notion to non-compact spaces as well. Theorem 6 thus claims that (M, ω, ϕ, J) is strongly cobordant to $\sum(\mathbb{C}^d, \omega_k, \phi_k, i)$ where the linear spaces are given their complex orientations. A stable complex G-manifold with a G-pre-quantum line bundle $\boldsymbol{L}$ can be made into a formal Hamiltonian G-space by replacing $\boldsymbol{L}$ by its G-equivariant first Chern class. With this definition we see that *the equivariant Riemann-Roch number* (20) *is an invariant of strong cobordism.*

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